

Observation problems posed for the Klein-Gordon equation

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Abstract. Transversal vibrations $u = u(x, t)$ of a string of length l with fixed ends are considered, where u is governed by the Klein-Gordon equation

$$u_{tt}(x, t) = a^2 u_{xx}(x, t) + cu(x, t), \quad (x, t) \in [0, l] \times \mathbb{R}, \quad a > 0, \quad c < 0.$$

Sufficient conditions are obtained that guarantee the solvability of each of four observation problems with given state functions f, g at two distinct time instants $-\infty < t_1 < t_2 < \infty$. The essential conditions are the following: smoothness of f, g as elements of a corresponding subspace $D^{s+i}(0, l)$ (introduced in [2]) of a Sobolev space $H^{s+i}(0, l)$, where $i = 1, 2$ depending on the type of the observation problem, and the representability of $t_2 - t_1$ as a rational multiple of $\frac{2l}{a}$. The reconstruction of the unknown initial data $(u(x, 0), u_t(x, 0))$ as the elements of $D^{s+1}(0, l) \times D^s(0, l)$ are given by means of the method of Fourier expansions.

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1. BACKGROUND AND KNOWN RESULTS

In control theory - which is closely related to the subject of this paper - numerous monographies and articles dealt with the accessibility of a final state (position and speed) of oscillations (in particular string oscillations) in the time interval $0 \leq t \leq T < \infty$; see for example, [1] - [10]. Although, only the short communication [11] dealt with observability of the string oscillations on the interval $0 \leq x \leq l$, and it treated just the case when the observation instants t_1 and t_2 are small, namely $0 \leq t_1 \leq t_2 \leq \frac{2l}{a}$, where a is the speed of the wave propagation. Furthermore, it is assumed in [11] that the initial data are known on some subinterval $[h_1, h_2] \subset [0, l]$. We reconstruct the initial data in each of the four observation problems related to the Klein-Gordon equation for arbitrary large t_1 and t_2 . Our preassumptions are only that $(t_2 - t_1)\frac{a}{2l}$ is rational and the given state functions are smooth enough. The cases $f, g \in D^s$ with arbitrary $s \in \mathbb{R}$ are also admitted.

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Let $\Omega = \{(x, t) : 0 < x < l, t \in \mathbb{R}\}$. Consider the problem (at first in the classical sense) of the vibrating $[0, l]$ string with fixed ends when there is an elastic withdrawing force proportional to the transversal deflection $u(x, t)$ of the point x of the string at the instant denoted by t . This phenomenon is described by the Klein-Gordon equation as follows:

$$(1) \quad u_{tt}(x, t) = a^2 u_{xx}(x, t) + cu(x, t), \quad (x, t) \in \overline{\Omega}, \quad a, c \in \mathbb{R}, \quad 0 < a, \quad 0 > c,$$

with the initial conditions

$$(2) \quad u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 \leq x \leq l,$$

and the homogeneous boundary conditions of the first kind

$$(3) \quad u(0, t) = 0, \quad u(l, t) = 0, \quad t \in \mathbb{R}.$$

We recall, that the function u is said to be a classical solution of this problem, if $u \in C^2(\overline{\Omega})$ and conditions (1) – (3) are satisfied.

It is well known that if

$$(4) \quad \varphi \in C^2[0, l], \psi \in C^1[0, l] \quad \text{and} \quad \varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \psi(0) = \psi(l) = 0,$$

then the Fourier method gives the classical solution u of the problem (1) – (3) posed for the Klein-Gordon equation, which is of the following form:

$$(5) \quad u(x, t) = \sum_{n=1}^{\infty} [\alpha_n \cos(t\omega_n) + \beta_n \sin(t\omega_n)] \sin\left(\frac{n\pi}{l}x\right), \quad (x, t) \in \overline{\Omega},$$

where

$$(6) \quad \omega_n = \sqrt{\left(\frac{n\pi}{l}a\right)^2 - c}, \quad n \in \mathbb{N},$$

$$(7) \quad \varphi(x) = u(x, 0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{l}x\right) \Rightarrow \alpha_n = \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi}{l}x\right) dx, \quad n \in \mathbb{N},$$

$$(8) \quad \psi(x) = u_t(x, 0) = \sum_{n=1}^{\infty} \omega_n \beta_n \sin\left(\frac{n\pi}{l}x\right) \Rightarrow \beta_n = \frac{1}{\omega_n} \frac{2}{l} \int_0^l \psi(x) \sin\left(\frac{n\pi}{l}x\right) dx, \quad n \in \mathbb{N}.$$

The uniqueness of the solution is a consequence of the law of conservation of energy.

To have a wider class of functions for φ, ψ and f, g , we shall consider certain generalized solutions of the problem (1)–(3). Namely, by using the suggestions of the referee,

we introduce the spaces $D^s(0, l)$, $s \in \mathbb{R}$ mentioned in the abstract (see [2]). Given an arbitrary real number s , on the linear span D of the functions $\sin \frac{n\pi}{l}x$, $n = 1, 2, \dots$, consider the following Euclidean norm:

$$\left\| \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right) \right\|_s := \left(\sum_{n=1}^{\infty} n^{2s} |c_n|^2 \right)^{\frac{1}{2}}.$$

Completing D with respect to this norm, we obtain a Hilbert space D^s . One can readily verify that for $s \geq 0$, D^s is a closed subspace of the Sobolev space $H^s(0, l)$, namely

$$D^s = \{u \in H^s(0, l) : u^{(2i)}(0) = u^{(2i)}(l) = 0, i = 0, 1, \dots, [(s-1)/2]\}.$$

If we identify $D^0 = L^2(0, l)$ with its dual, then D^{-s} is the dual space of D^s . Some of the results of [2] (see Section 1.1-1.3) and [10] say that for arbitrary $s \in \mathbb{R}$ with $(\varphi, \psi) \in D^{s+1} \times D^s$ the generalized mixed problem (1) – (3) has a unique solution u satisfying

$$u \in C(\mathbb{R}, D^{s+1}) \cap C^1(\mathbb{R}, D^s) \cap C^2(\mathbb{R}, D^{s-1})$$

given by the Fourier series (5) with coefficients α_n, β_n defined by (7) and (8). Here and below all Fourier expansions for φ, ψ, f, g and u are understood in the spaces $D^s(0, l)$.

2. NEW RESULTS

Definition 1. The observation problem posed for the Klein-Gordon equation is the following. The initial functions φ, ψ are unknown, but such functions $f(x)$ and $g(x)$ are given for which one of the following four conditions holds:

$$(9) \quad u(x, t_1) = f(x), \quad u(x, t_2) = g(x), \quad 0 \leq x \leq l;$$

$$(10) \quad u_t(x, t_1) = f(x), \quad u(x, t_2) = g(x), \quad 0 \leq x \leq l;$$

$$(11) \quad u(x, t_1) = f(x), \quad u_t(x, t_2) = g(x), \quad 0 \leq x \leq l;$$

$$(12) \quad u_t(x, t_1) = f(x), \quad u_t(x, t_2) = g(x), \quad 0 \leq x \leq l.$$

Here u is the solution of the generalized problem (1) – (3), and the given functions f, g are said to be the partial state of the string at distinct time instants t_1 and t_2 , $-\infty < t_1 < t_2 < \infty$. Now the problem is to find the initial functions φ, ψ in terms of $f(x), g(x)$.

Theorem 1. *Suppose that*

$$(13) \quad f \in D^{s+2}, \quad g \in D^{s+2}, \quad \text{where } s \in \mathbb{R},$$

$$(14) \quad t_2 - t_1 = \frac{p}{q} \frac{2l}{a},$$

where p, q are positive integers and they are relative primes. In addition, suppose that

$$(15) \quad \sin \left((t_2 - t_1) \sqrt{\left(\frac{n\pi}{l}a\right)^2 - c} \right) \neq 0, \quad \forall n \in \mathbb{N}.$$

Then the observation problem (1) – (3) under condition (9) has a unique solution for $(\varphi, \psi) \in D^{s+1} \times D^s$. They are represented by their Fourier expansions in the proof below.

Theorem 2. *Suppose that*

$$(16) \quad f \in D^{s+1}, \quad g \in D^{s+2}, \quad \text{where } s \in \mathbb{R},$$

condition (14) holds and

$$(17) \quad \cos \left((t_2 - t_1) \sqrt{\left(\frac{n\pi}{l}a\right)^2 - c} \right) \neq 0, \quad \forall n \in \mathbb{N}.$$

Then the observation problem (1) – (3) under condition (10) has a unique solution for $(\varphi, \psi) \in D^{s+1} \times D^s$. They are represented by their Fourier expansions in the proof below.

Theorem 3. *Suppose that*

$$(18) \quad f \in D^{s+2}, \quad g \in D^{s+1}, \quad \text{where } s \in \mathbb{R},$$

and conditions (14) and (17) hold. Then the observation problem (1) – (3) under condition (11) has a unique solution for $(\varphi, \psi) \in D^{s+1} \times D^s$. They are represented by their Fourier expansions in the proof below.

Theorem 4. *Suppose that*

$$(19) \quad f \in D^{s+1}, \quad g \in D^{s+1}, \quad \text{where } s \in \mathbb{R},$$

and conditions (14) and (15) hold. Then the observation problem (1) – (3) under condition (12) has a unique solution for $(\varphi, \psi) \in D^{s+1} \times D^s$. They are represented by their Fourier expansions in the proof below.

3. AUXILIARY RESULTS

Lemma 1. *If condition (14) holds, then there exist $N \in \mathbb{N}$ and $m \in \mathbb{R}$ such that*

$$\frac{1}{|\sin(\omega_n(t_2 - t_1))|} < \frac{n}{m}, \quad \forall n > N.$$

Proof. First, we deal with the denominator of the left-hand side of the inequality

$$\begin{aligned} (20) \quad \sin(\omega_n(t_2 - t_1)) &= \sin\left((t_2 - t_1)\frac{n\pi}{l}a + (t_2 - t_1)\left[\omega_n - \frac{n\pi}{l}a\right]\right) = \\ &= \sin\left((t_2 - t_1)\frac{n\pi}{l}a + (t_2 - t_1)\frac{\omega_n^2 - (\frac{n\pi}{l}a)^2}{\omega_n + \frac{n\pi}{l}a}\right) = \sin\left((t_2 - t_1)\frac{n\pi}{l}a + (t_2 - t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right). \end{aligned}$$

It follows from the condition (14) that

$$(t_2 - t_1)\frac{n\pi}{l}a = \frac{p}{q}2n\pi,$$

and that it takes on at most q different values (mod 2π) as n varies. Let

$$z_n := (t_2 - t_1)\frac{n\pi}{l}a \quad \text{and} \quad d_1 := \min_{n, \sin z_n \neq 0} \{|\sin(z_n)|\}.$$

Due to the absolute value bars, there is a real number d_2 such that

$$\sin(d_2) = d_1, \quad 0 < d_2 \leq \frac{\pi}{2}.$$

It is easy to see, that

$$(t_2 - t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a} = O\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty.$$

Therefore, there exist constants $N \in \mathbb{N}$, $m \in \mathbb{R}^+$ such that

$$(21) \quad \frac{\pi m}{2n} < \left| (t_2 - t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a} \right| < \frac{d_2}{2} \quad \text{and} \quad \frac{m}{n} < \sin\left(\frac{d_2}{2}\right), \quad \forall n > N.$$

So, if $\sin\left((t_2 - t_1)\frac{n\pi}{l}a\right) \neq 0$, then

$$\left| \sin\left((t_2 - t_1)\frac{n\pi}{l}a + (t_2 - t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right) \right| > \left| \sin\left(d_2 - \frac{d_2}{2}\right) \right| = \sin\left(\frac{d_2}{2}\right) > \frac{m}{n},$$

whenever $n > N$, by virtue of (21).

On the other hand, if $\sin\left((t_2 - t_1)\frac{n\pi}{l}a\right) = 0$, then

$$\begin{aligned} &\left| \sin\left((t_2 - t_1)\frac{n\pi}{l}a + (t_2 - t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right) \right| = \left| \sin\left((t_2 - t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a}\right) \right| > \\ &> \frac{2}{\pi} \left| (t_2 - t_1)\frac{-c}{\omega_n + \frac{n\pi}{l}a} \right| > \frac{m}{n}, \quad \forall n > N, \end{aligned}$$

due to (21) and the inequality

$$(22) \quad |\sin t| > \frac{2}{\pi} |t|, \quad \text{if } 0 < |t| < \frac{\pi}{2}.$$

Combining the two cases just above, we get that

$$\left| \sin \left((t_2 - t_1) \frac{n\pi}{l} a + (t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right) \right| > \frac{m}{n}, \quad \forall n > N.$$

□

Lemma 2. *If condition (14) holds, then there exist $N \in \mathbb{N}$ and $m \in \mathbb{R}$ such that*

$$\frac{1}{|\cos(\omega_n(t_2 - t_1))|} < \frac{n}{m}, \quad \forall n > N.$$

Proof. Similarly to (20) in the proof of Lemma 1, now we obtain that

$$\cos(\omega_n(t_2 - t_1)) = \cos \left((t_2 - t_1) \frac{n\pi}{l} a + (t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right).$$

Let

$$z_n := (t_2 - t_1) \frac{n\pi}{l} a \quad \text{and} \quad d_1 := \min_{n, \cos z_n \neq 0} \{|\cos z_n|\}.$$

Due to the absolute value bars, there is a real number d_2 such that

$$\cos(d_2) = d_1, \quad 0 \leq d_2 < \frac{\pi}{2}.$$

Similarly to (21) in the proof of Lemma 1, there exist constants $N \in \mathbb{N}$ and $m \in \mathbb{R}^+$ such that

$$(23) \quad \frac{\pi m}{2n} < \left| (t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right| < \frac{\frac{\pi}{2} - d_2}{2} \quad \text{and} \quad \frac{m}{n} < \cos \left(\frac{\frac{\pi}{2} + d_2}{2} \right), \quad \forall n > N.$$

In this manner, if $\cos \left((t_2 - t_1) \frac{n\pi}{l} a \right) \neq 0$, we obtain again that

$$\left| \cos \left((t_2 - t_1) \frac{n\pi}{l} a + (t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right) \right| > \left| \cos \left(d_2 + \frac{\frac{\pi}{2} - d_2}{2} \right) \right| = \cos \left(\frac{\frac{\pi}{2} + d_2}{2} \right) > \frac{m}{n},$$

whenever $n > N$, by virtue of (23).

On the other hand, in the case when $\cos \left((t_2 - t_1) \frac{n\pi}{l} a \right) = 0$, we get

$$\begin{aligned} & \left| \cos \left((t_2 - t_1) \frac{n\pi}{l} a + (t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right) \right| = \left| \sin \left((t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right) \right| > \\ & > \frac{2}{\pi} \left| (t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right| > \frac{m}{n}, \quad \forall n > N, \end{aligned}$$

due to (22) and (23).

Combining the two cases just above, we get that

$$\left| \cos \left((t_2 - t_1) \frac{n\pi}{l} a + (t_2 - t_1) \frac{-c}{\omega_n + \frac{n\pi}{l} a} \right) \right| > \frac{m}{n}, \quad \forall n > N.$$

□

4. PROOFS OF THE THEOREMS 1 – 4

Proof of Theorem 1. Since any of the solutions u of problem (1)–(3) has representation (5) with some coefficients α_n, β_n ; $n \in \mathbb{N}$, the observation problem can be reduced to the problem of the appropriate choices of α_n and β_n such that (9) is satisfied. For this reason, we substitute t_1 and t_2 into (5), and use the two conditions in (9). As a result, we get the following necessary conditions for α_n, β_n :

$$(24) \quad f(x) = u(x, t_1) = \sum_{n=1}^{\infty} [\alpha_n \cos(\omega_n t_1) + \beta_n \sin(\omega_n t_1)] \sin\left(\frac{n\pi}{l} x\right), \quad x \in [0, l],$$

$$(25) \quad g(x) = u(x, t_2) = \sum_{n=1}^{\infty} [\alpha_n \cos(\omega_n t_2) + \beta_n \sin(\omega_n t_2)] \sin\left(\frac{n\pi}{l} x\right), \quad x \in [0, l],$$

where ω_n is defined in (6).

The assumption (13) guarantees that the coefficients of the sine Fourier expansions of the functions $f(x), g(x)$ are unambiguously determined and comparing these Fourier series with (24) and (25), for α_n, β_n we get the following conditions:

$$(26) \quad \begin{aligned} \alpha_n \cos(\omega_n t_1) + \beta_n \sin(\omega_n t_1) &= \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx, \quad n \in \mathbb{N}, \\ \alpha_n \cos(\omega_n t_2) + \beta_n \sin(\omega_n t_2) &= \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l} x\right) dx, \quad n \in \mathbb{N}. \end{aligned}$$

The linear system (26) can be uniquely solved for the unknown coefficients α_n and β_n due to assumption (15):

$$(27) \quad \begin{aligned} \alpha_n &= \frac{\sin(\omega_n t_2) \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx - \sin(\omega_n t_1) \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l} x\right) dx}{\sin(\omega_n(t_2 - t_1))}, \\ \beta_n &= \frac{-\cos(\omega_n t_2) \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx + \cos(\omega_n t_1) \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l} x\right) dx}{\sin(\omega_n(t_2 - t_1))}. \end{aligned}$$

So the unknown initial functions φ and ψ are uniquely determined and found in the form of (7) and (8). It remains to show that φ, ψ are from the classes D^{s+1}, D^s , respectively, i. e. to show that the following inequality holds:

$$(28) \quad \max\{\|\varphi\|_{s+1}^2, \|\psi\|_s^2\} < \infty.$$

We introduce the following notations for the sake of transparency:

$$D_n := \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx,$$

$$E_n := \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) dx.$$

Since $(f, g) \in D^{s+2} \times D^{s+2}$, we have the following inequality:

$$(29) \quad \sum_{n=1}^{\infty} n^{2s+4} \max\{|D_n|^2, |E_n|^2\} < \infty.$$

By using Lemma 1, for every $n > N$ we get

$$|\alpha_n| = \left| \frac{\sin(\omega_n t_2) D_n - \sin(\omega_n t_1) E_n}{\sin(\omega_n(t_2 - t_1))} \right| < \left| \frac{n}{m} D_n \right| + \left| \frac{n}{m} E_n \right|,$$

$$|\beta_n| = \left| \frac{-\cos(\omega_n t_2) D_n + \cos(\omega_n t_1) E_n}{\sin(\omega_n(t_2 - t_1))} \right| < \left| \frac{n}{m} D_n \right| + \left| \frac{n}{m} E_n \right|,$$

which means that

$$(30) \quad \max\{|\alpha_n|, |\beta_n|\} < c_1 n \max\{|D_n|, |E_n|\} \quad n \in \mathbb{N},$$

with a suitable constant c_1 .

Let $M \geq 1$ be a constant such that $\omega_n < Mn$, $\forall n \in \mathbb{N}$. Combining (29), (30) and the definition of the norm $\|\cdot\|_s$ we get the desired inequality (28):

$$\begin{aligned} \max\{\|\varphi\|_{s+1}^2, \|\psi\|_s^2\} &= \max\left\{\sum_{n=1}^{\infty} n^{2s+2} |\alpha_n|^2, \sum_{n=1}^{\infty} n^{2s} |\omega_n \beta_n|^2\right\} \leq \\ &\leq \sum_{n=1}^{\infty} M^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} < c_1^2 M^2 \sum_{n=1}^{\infty} n^{2s+4} \max\{|D_n|^2, |E_n|^2\} < \infty. \end{aligned}$$

□

Remark 1. In the classical case when the given state functions are continuously differentiable, according to Theorem 1, the initial functions are also continuously differentiable. More precisely, if

$$u(x, t_1) = f(x) \in C^4[0, l], \quad u(x, t_2) = g(x) \in C^4[0, l], \quad f, g|_{0,l} = f'', g''|_{0,l} = 0,$$

then $f, g \in D^4$ and the observation problem has a unique classical solution

$$u(x, 0) = \varphi(x) \in D^3 \subset C^2, \quad u_t(x, 0) = \psi(x) \in D^2 \subset C^1.$$

Remark 2. Taking into account (20), condition (15) can be written into the following form:

$$(31) \quad \sin((t_2 - t_1)\omega_n) = \sin\left((t_2 - t_1)\frac{n\pi}{l}a + (t_2 - t_1)\frac{-c}{\sqrt{(\frac{n\pi}{l}a)^2 - c} + \frac{n\pi}{l}a}\right) \neq 0$$

for all $n \in \mathbb{N}$. Analysing the proof of Lemma 1, it is easy to see that the above condition is certainly satisfied for all n large enough, say $n > N$.

If we want to get an easily verifiable condition instead of (15), which is not necessary then

$$(32) \quad (t_2 - t_1)\frac{-c}{\sqrt{(\frac{\pi}{l}a)^2 - c} + \frac{\pi}{l}a} < \frac{\pi}{q}$$

is such a sufficient condition. We justify this claim as follows. The first term in the argument of the sine function in (31) is either 0 (mod 2π), or its distance is at least $\frac{\pi}{q}$ from its zeroes, and the second term in the argument of the sine function in (31) is positive and monotone decreasing function of n . So, if we assume that the second term is already smaller than $\frac{\pi}{q}$ for $n = 1$, which is actually the case in (32), then condition (31) is satisfied for each $n \geq 1$.

Nevertheless, we can see from this simpler condition (32), that if the parameters $|c|$ and a in equation (1) are such that either c is small or a is great enough, then condition (31) is always satisfied. Similar observations can be made in the following Theorems 2 – 4.

Proof of Theorem 2. In an analogous way as in the proof of Theorem 1, now we start with the following equalities:

$$f(x) = u_t(x, t_1) = \sum_{n=1}^{\infty} [-\alpha_n \omega_n \sin(\omega_n t_1) + \beta_n \omega_n \cos(\omega_n t_1)] \sin\left(\frac{n\pi}{l}x\right), \quad x \in [0, l],$$

$$g(x) = u(x, t_2) = \sum_{n=1}^{\infty} [\alpha_n \cos(\omega_n t_2) + \beta_n \sin(\omega_n t_2)] \sin\left(\frac{n\pi}{l}x\right), \quad x \in [0, l].$$

Hence we get the following necessary conditions for the coefficients α_n, β_n :

$$-\alpha_n \omega_n \sin(\omega_n t_1) + \beta_n \omega_n \cos(\omega_n t_1) = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx, \quad n \in \mathbb{N},$$

$$\alpha_n \cos(\omega_n t_2) + \beta_n \sin(\omega_n t_2) = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) dx, \quad n \in \mathbb{N}.$$

The linear equations just received can be uniquely solved for the unknown coefficients α_n and β_n , due to assumption (17):

$$\alpha_n = \frac{-\sin(\omega_n t_2) \frac{2}{l} \int_0^l f(x) \sin(\frac{n\pi}{l} x) dx + \cos(\omega_n t_1) \omega_n \frac{2}{l} \int_0^l g(x) \sin(\frac{n\pi}{l} x) dx}{\omega_n \cos(\omega_n(t_2 - t_1))},$$

$$\beta_n = \frac{\cos(\omega_n t_2) \frac{2}{l} \int_0^l f(x) \sin(\frac{n\pi}{l} x) dx + \sin(\omega_n t_1) \omega_n \frac{2}{l} \int_0^l g(x) \sin(\frac{n\pi}{l} x) dx}{\omega_n \cos(\omega_n(t_2 - t_1))}.$$

So the unknown initial functions φ and ψ are uniquely determined and found in the form of (7) and (8). It remains to show that φ , ψ are from the classes D^{s+1} , D^s , respectively. To this effect, it is enough to show that (28) holds.

Again, let

$$D_n := \frac{2}{l} \int_0^l f(x) \sin(\frac{n\pi}{l} x) dx,$$

$$E_n := \frac{2}{l} \int_0^l g(x) \sin(\frac{n\pi}{l} x) dx.$$

Since $(f, g) \in D^{s+1} \times D^{s+2}$, we have that the inequality (29') holds:

$$(29') \quad \sum_{n=1}^{\infty} n^{2s+4} \max\{|\frac{1}{n} D_n|^2, |E_n|^2\} < \infty.$$

By using Lemma 2, for every $n > N$ we have

$$|\alpha_n| = \left| \frac{-\sin(\omega_n t_2) D_n + \cos(\omega_n t_1) \omega_n E_n}{\omega_n \cos(\omega_n(t_2 - t_1))} \right| < \left| \frac{1}{\omega_n} \frac{n}{m} D_n \right| + \left| \frac{n}{m} E_n \right|,$$

$$|\beta_n| = \left| \frac{\cos(\omega_n t_2) D_n + \sin(\omega_n t_1) \omega_n E_n}{\omega_n \cos(\omega_n(t_2 - t_1))} \right| < \left| \frac{1}{\omega_n} \frac{n}{m} D_n \right| + \left| \frac{n}{m} E_n \right|,$$

which means that

$$(30') \quad \max\{|\alpha_n|, |\beta_n|\} < c_2 n \max\{|\frac{1}{n} D_n|, |E_n|\} \quad n \in \mathbb{N},$$

with a suitable constant c_2 .

Combining (29'), (30') and the definition of the norm $\|\cdot\|_s$ we get the desired inequality (28):

$$\begin{aligned} \max\{\|\varphi\|_{s+1}^2, \|\psi\|_s^2\} &= \max\left\{\sum_{n=1}^{\infty} n^{2s+2} |\alpha_n|^2, \sum_{n=1}^{\infty} n^{2s} |\omega_n \beta_n|^2\right\} \leq \\ &\leq \sum_{n=1}^{\infty} M^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} < c_2^2 M^2 \sum_{n=1}^{\infty} n^{2s+4} \max\{|\frac{1}{n} D_n|^2, |E_n|^2\} < \infty. \end{aligned} \quad \square$$

Proof of Theorem 3. This proof goes along the same lines as that of Theorem 2, except that here we have to interchange the roles of the coefficients α_n and β_n . \square

Proof of Theorem 4. Now, we have

$$f(x) = u_t(x, t_1) = \sum_{n=1}^{\infty} [-\alpha_n \omega_n \sin(\omega_n t_1) + \beta_n \omega_n \cos(\omega_n t_1)] \sin\left(\frac{n\pi}{l}x\right), \quad x \in [0, l],$$

$$g(x) = u_t(x, t_2) = \sum_{n=1}^{\infty} [-\alpha_n \omega_n \sin(\omega_n t_2) + \beta_n \omega_n \cos(\omega_n t_2)] \sin\left(\frac{n\pi}{l}x\right), \quad x \in [0, l],$$

whence the necessary conditions for the coefficients α_n, β_n are the following:

$$-\alpha_n \omega_n \sin(\omega_n t_1) + \beta_n \omega_n \cos(\omega_n t_1) = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx, \quad n \in \mathbb{N},$$

$$-\alpha_n \omega_n \sin(\omega_n t_2) + \beta_n \omega_n \cos(\omega_n t_2) = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) dx, \quad n \in \mathbb{N}.$$

The linear equations just received can be uniquely solved for the unknown coefficients α_n and β_n , due to assumption (15):

$$\alpha_n = \frac{\cos(\omega_n t_2) \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx - \cos(\omega_n t_1) \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) dx}{\omega_n \sin(\omega_n(t_2 - t_1))},$$

$$\beta_n = \frac{\sin(\omega_n t_2) \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx - \sin(\omega_n t_1) \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) dx}{\omega_n \sin(\omega_n(t_2 - t_1))}.$$

So the unknown initial functions φ and ψ are uniquely determined and found in the form of (7) and (8). It remains to show that φ, ψ are from the classes D^{s+1}, D^s , respectively. To this effect, it is enough to show that (28) holds.

Again, let

$$D_n := \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx,$$

$$E_n := \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi}{l}x\right) dx.$$

Since $(f, g) \in D^{s+1} \times D^{s+1}$, we have that the inequality (29'') holds:

$$(29'') \quad \sum_{n=1}^{\infty} n^{2s+2} \max\{|D_n|^2, |E_n|^2\} < \infty.$$

By using Lemma 1, for every $n > N$ we get

$$|\alpha_n| = \left| \frac{\cos(\omega_n t_2) D_n - \cos(\omega_n t_1) E_n}{\omega_n \sin(\omega_n(t_2 - t_1))} \right| < \left| \frac{1}{\omega_n} \frac{n}{m} D_n \right| + \left| \frac{1}{\omega_n} \frac{n}{m} E_n \right|,$$

$$|\beta_n| = \left| \frac{\sin(\omega_n t_2) D_n - \sin(\omega_n t_1) E_n}{\omega_n \sin(\omega_n(t_2 - t_1))} \right| < \left| \frac{1}{\omega_n} \frac{n}{m} D_n \right| + \left| \frac{1}{\omega_n} \frac{n}{m} E_n \right|,$$

which means that

$$(30'') \quad \max\{|\alpha_n|, |\beta_n|\} < c_4 \max\{|D_n|, |E_n|\} \quad n \in \mathbb{N},$$

with a suitable constant c_4 .

Combining (29''), (30'') and the definition of the norm $\|\cdot\|_s$ we get the desired inequality (28):

$$\begin{aligned} \max\{\|\varphi\|_{s+1}^2, \|\psi\|_s^2\} &= \max\left\{\sum_{n=1}^{\infty} n^{2s+2} |\alpha_n|^2, \sum_{n=1}^{\infty} n^{2s} |\omega_n \beta_n|^2\right\} \leq \\ &\leq \sum_{n=1}^{\infty} M^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} < c_4^2 M^2 \sum_{n=1}^{\infty} n^{2s+2} \max\{|D_n|^2, |E_n|^2\} < \infty. \end{aligned} \quad \square$$

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